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REGIONS OF ASYMPTOTIC STABILITY OF LIÉNARD'S EQUATION

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We consider the Liénard equation

$$x^{\cdot\cdot} + f(x)x^{\cdot} + g(x) = 0 \tag{1}$$

We assume the functions $f(x)$ and $g(x)$ to be such, that

a) the conditions of existence and uniqueness of the solution are fulfilled,

b) $xg(x) > 0$ for any $x \in (a, b)$ and $x \neq 0$,

c) $x \int_0^x f(x)\, dx > 0$ for any $x \in [a, b]$ and $x \neq 0$, and,

that $x \in (-x_m, x_m) \subset [a, b]$ can be found such that $f(x) > 0$.

A system equivalent to (1) has the form

$$x^{\cdot} = y, \quad y^{\cdot} = -f(x)\, y - g(x) \tag{2}$$

The stability of the origin of coordinates was studied for more general systems using the Liapunov functions, in [1 – 5] and others.

The present paper introduces the additional condition (c), which makes possible a simple estimation of the region of attraction of the origin of coordinates. The result obtained is used to study the systems encountered in radio engineering and automation.

Theorem. If conditions (a), (b) and (c) hold for system (2), then the equilibrium position $x = y = 0$ of the system is asymptotically stable. The region of attraction of zero solution is obtained in the form

$$\frac{1}{2} y^2 = \int_x^{x_i} g(x)\, dx = G(x_i) - G(x) \tag{3}$$

The upper limit of integration x_i depends on the behavior of the functions $f(x)$ and $g(x)$ on the interval (a, b). Thus, in (2), the root of the equation $f(x) = 0$, $x \in [a, b]$ nearest to the origin of coordinates is chosen as $x_i = x_m$. In the case when the equation $f(x) = 0$ has no roots on the interval $[a, b]$, it is assumed that $x_m = \min(|a|, |b|)$. Passing to the Liénard phase plane variables

$$x^{\cdot} = y - F(x), \quad y^{\cdot} = -g(x), \quad F(x) = \int_0^x f(x)\, dx \tag{4}$$

we use the root of the equation $g(x) = 0$ nearest to the origin of coordinates as $x_i = x_k$.

To prove the theorem, we consider the Liapunov function in the form

$$V(x, y) = \frac{1}{2} y^2 + \int_0^x g(x)\, dx$$

which, by virtue of the condition (b) of the theorem, is positive definite in the region $D: \{x \in (a, b),\ x \neq 0,\ |y| < +\infty\}$. Taking the total derivative of V with respect to time we obtain, by virtue of the system (2) and therefore of the system (4),

$$V^{\cdot}|_{(2)} = -y^2 f(x), \qquad V^{\cdot}|_{(4)} = -g(x) F(x)$$

Under the restrictions (a), (b) and (c) formulated above we have $V^{\cdot}|_{(2)} < 0$ and $V^{\cdot}|_{(4)} < 0$ for all $x, y \in D$. From the systems (2) and (4) it follows directly that the sets $V^{\cdot}|_{(2)} = 0 (y = 0)$ and $V^{\cdot}|_{(5)} = 0$ (either $g(x) = 0$ or $F(x) = 0$) do not contain complete trajectories, the exception being the state of equilibrium $x = y = 0$. Then by the theorems of [1, 2], the state of equilibrium $x = y = 0$ is asymptotically stable.

In the region (3) the following inequality holds for all $x, y \in D$

$$0 < V(x, y) < V(x_i, y)$$

and this implies [2 – 4] that (3) is the region of attraction of the origin of coordinates.

We shall use the results obtained to investigate the systems encountered in radio engineering and automation [6–8]. For example, let $f(x) = (a / \alpha) g'(x)$, where $a, \alpha > 0$ [8], and let $g(x)$ be an odd function such that

$$g(x + 2\pi) = g(x), \qquad g(0) = g(\pi) = g(-\pi) = 0$$

$$\max_{x \in (-\pi, \pi)} g(x) = -\min_{x \in (-\pi, \pi)} g(x) = 1$$

In this example $x_m = x_i$ is found from the equation $g'(x) = 0$, and $x_i = x_k$ is the root of the equation $g(x) = 0$ nearest to the origin. The case corresponds to an astatic control system. As another example [7] we consider Eq. (1) with

$$f(x) = \alpha [1 + \lambda g_1'(x)], \; g(x) = g_1(x) - \beta, \; \alpha, \lambda, \beta > 0 \tag{5}$$

The function $g_1(x)$ here is identical with that of the previous example. The abscissae of the state of equilibrium are defined from the equation

$$g_1(x) = \beta \quad (g(x) = 0)$$

Let $(x_0, 0)$ be the stable state of equilibrium. We transfer the origin of coordinates to the point $(x_0, 0)$, setting $x = \varphi + x_0$. In the case (5) Eq. (1) assumes the form

$$\varphi^{\cdot\cdot} + f(\varphi) \varphi^{\cdot} + g(\varphi) = 0 \tag{6}$$

$$f(\varphi) = \alpha [1 + \lambda g_1'(\varphi + x_0)], \; g(\varphi) = g_1(\varphi + x_0) - \beta$$

Equation (6) is considered in the strip $L: \{x_1 - x_0 - 2\pi < \varphi < x_1 - x_0, \; |y| < +\infty\}$, where x_1 is the abscissa of the unstable state of equilibrium satisfying the assertions formulated above. The upper bound for Eq. (6) is

$$\varphi_i = \begin{cases} \varphi_m \leqslant \varphi_1 & \text{in the case (2)} \\ \varphi_1 = x_1 - x_0 & \text{in the case (4)} \end{cases}$$

and φ_m is defined from

$$g_1'(\varphi + x_0) = -\lambda^{-1} \tag{7}$$

This implies that (7) has a solution only when $\lambda \geqslant \lambda_*$ where $\lambda_* = |g_{\max}|^{-1}$; when $\lambda < \lambda_*$ we have, by virtue of the system (2) $V^{\cdot} < 0$ for any $-\varphi_1 < \varphi < \varphi_1$ since $f(\varphi) > 0$ for $\varphi \in (-\varphi_1, \varphi_1)$.

Thus the region of asymptotic stability becomes narrower with increasing parameter λ and in the limit as $\lambda \to \infty$ is defined by the equation $g'(\varphi + x_0) = 0$.

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ON WAVE FIELDS AND ACUTE-ANGLED EDGES ON WAVE FRONTS IN AN ANISOTROPIC MEDIUM FROM A POINT SOURCE

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In addition to the results in [1], wave fields of quasi-longitudinal and quasi-transverse elastic vibrations from a point source of an instantaneous pulse type are studied in an anisotropic medium with four elastic constants. Cases are considered when the wave fronts are convex closed curves and when the inner front consists of piecewise-smooth curves forming acute-angled edges.

The solution characterizing the elastic vibrations of quasi-longitudinal and quasi-transverse type SV waves in an infinite anisotropic medium from a point source of instantaneous pulse type placed at the origin is [1]

$$u = \sum_{k=1}^{2} R\left\{c\int^{\theta_k} \zeta\lambda_k w_k(\zeta)\, d\zeta\right\} \tag{1}$$

$$v = \sum_{k=1}^{2} R\left\{\int^{\theta_k} (a\zeta^2 + d\lambda_k^2 - 1)\, w_k(\zeta)\, d\zeta\right\}$$

The complex variables θ_k and the quantities λ_k are defined by the following relationsips: